Two-state teleportation

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Quantum teleportation with additional *a priori* information about the input state achieves higher fidelity than teleportation of a completely unknown state. However, perfect teleporation of two nonorthogonal input states requires the same amount of entanglement as perfect teleportation of an unknown state, namely one ebit. We analyze how well two-state teleportation can be achieved using every degree of pure-state entanglement. We find the highest fidelity of ''teleportation'' that can be achieved with only classical communication but no shared entanglement. A two-state telecloning scheme is constructed.

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I. INTRODUCTION

The transmission of quantum states can be accomplished by either the direct sending of qubits or by the transmission of classical bits where the sender and receiver share entanglement. In schemes for quantum teleportation, it has been shown that the transmission of two classical bits, together with the use of one ebit, achieves the same results as sending one qubit [1].

If the state to be teleported is completely unknown, the fact that the amount of entanglement between two separated subsystems may not increase under local operations means that faithful teleporation cannot be achieved without one full unit of entanglement. The argument goes as follows. Alice's particle is initially in an unknown state, which could be a mixed state due to entanglement with another particle R at Alice's end. After the teleportation, the entanglement between Alice's particle and R is transferred to an entanglement between Bob's particle and R by entanglement swapping [2]. The original entangled channel between Alice and Bob is completely destroyed. Local operations and classical communication cannot increase the entanglement between Alice and Bob. Therefore the original entanglement in the channel must be at least as high as the final entanglement between Bob's particle and R. However, the initial state of Alice's particle is completely unknown; it may be a maximally mixed state, arising because Alice's particle is maximally entangled to another particle, R. This would make the final entanglement between Bob and R maximal. Therefore the initial entanglement in the channel must also be maximal [3].

On the other hand, if Alice knows exactly what state she has, there is no need for any entanglement to reliably transmit the state. She simply sends Bob classical information saying which state it is, and he prepares it himself.

Between the two extremes of Alice possessing no prior information of the state and Alice possessing full information, she may have some prior knowledge. For example, she may receive her qubits from a known ensemble $\epsilon =\{|\phi_x\rangle,p_x\}$ of states $|\phi_x\rangle$ with probability p_x. We consider the situation where Alice knows that a preparer of quantum states provides her with one of two nonorthogonal states, say, $|\psi_1\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$ and $|\psi_2\rangle=\sin(\theta/2)|0\rangle+\cos(\theta/2)|1\rangle$ with equal probabilities. Alice then knows almost everything about the state. In effect, she has to transmit one bit of information to Bob saying which of the two states she has. Is it possible to teleport the quantum state in this case, using less than the full unit of entanglement required when the state is completely unknown?

It turns out, rather surprisingly, that it is not possible and that a full unit of entanglement is needed even for teleportation of only two states. This is shown in Sec. III. In Sec. II, we find the upper bound for the fidelity of sending the state with no entanglement. In Sec. IV, we consider teleportation using a nonmaximally entangled channel. We make some connections between teleportation and cloning in Sec. V and adapt the telecloning scheme of Murao *et al.* [4] to the case of telecloning two nonorthogonal states. The two-state telecloning state is now different from that for universal telecloning. We find that the amount of entanglement required between the sender and recipients is now state dependent.

II. SCHEMES WITHOUT ENTANGLEMENT

For comparison, we first determine what fidelity of transmission can be achieved without using any entanglement, only classical communication. Alice measures her state and sends the result to Bob, who makes his best guess of the state based on this information. The fidelity of sending the state $|\psi\rangle$ is defined as

$$F_{cl}(|\psi\rangle)=\sum_{i=1}^{n} P(i|\psi)|\langle\psi|\alpha_i\rangle|^2, \qquad (1)$$

where $P(i|\psi)=\langle\psi|A_i|\psi\rangle$ is the probability of Alice obtaining the result corresponding to the positive operator A_i out of n possible outcomes of the positive operator valued measure (POVM) $\{A_i\}$ where $\Sigma_{i=1}^{n}A_i=1$. The state $|\alpha_i\rangle$ is Bob's guess, given outcome i.

When the input state is completely unknown, the average of the fidelity over an even distribution of all states on the Bloch sphere is taken. It has been shown that the average fidelity over all states is [5]

$$F_{cl}=\int\sum_{i=1}^{n} P[(i|\psi)|\langle\psi|\alpha_i\rangle|^2]d\Omega=\tfrac{2}{3}.$$

In this case, Alice may make an orthogonal measurement in any direction, and it is optimal for Bob to prepare the state corresponding to Alice's result.

On the other hand, when Alice's state is drawn from an ensemble of two states, $\{|\psi_1\rangle,|\psi_2\rangle\}$, with equal probabilities, the fidelity

$$F_{cl}(\{|\psi_1\rangle,|\psi_2\rangle\})=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{2} P(i|\psi_j)|\langle\psi_j|\alpha_i\rangle|^2$$

is much higher. This is the case we consider in this paper.

We first calculate the fidelity in the case where Bob simply prepares a guessed state corresponding to one of the two input states, $|\alpha_i\rangle\in\{|\psi_1\rangle,|\psi_2\rangle\}$, for all $i=1,\ldots,n$. Then the fidelity is limited only by the errors Alice makes in measuring, due to the fact that the signal states are nonorthogonal. We employ previous results on distinguishing two states. These results have been derived with respect to two different ways of characterizing distinguishability. The states may either be distinguished so as to minimize the probability of error in guessing the right state, or by using an ''unambiguous'' measurement, which has no probability of error, but which sometimes yields no information about the state.

It has been shown [6] that the smallest attainable probability of error in distinguishing two states is

$$P_e=\tfrac{1}{2}-\tfrac{1}{4}\,\mathrm{Tr}(|\rho_1-\rho_0|).$$

For two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, the minimal probability of error may be derived from the unitary evolution of the unknown state and an ancilla qubit, initially in the state $|0\rangle_A$, on which a projective measurement will be performed in the $\{|0\rangle,|1\rangle\}$ basis:

$$|0\rangle_A|\psi_1\rangle\rightarrow\sqrt{1-P_e}|0\rangle_A|\psi_1\rangle+\sqrt{P_e}|1\rangle_A|\psi_2\rangle,$$
$$|0\rangle_A|\psi_2\rangle\rightarrow\sqrt{P_e}|0\rangle_A|\psi_1\rangle+\sqrt{1-P_e}|1\rangle_A|\psi_2\rangle. \quad (2)$$

If the ancilla is measured in the state $|0\rangle_A$, we conclude that the state is $|\psi_1\rangle$, and if $|1\rangle_A$, then we conclude $|\psi_2\rangle$. The requirement that this evolution be unitary gives

$$P_e=\tfrac{1}{2}(1\pm\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2}).$$

For two pure states $|\psi_1\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$ and $|\psi_2\rangle=\sin(\theta/2)|0\rangle+\cos(\theta/2)|1\rangle$, this is given by

$$P_e=\tfrac{1}{2}(1-\cos\theta).$$

If $\theta=0$, the two states are orthogonal and the probability of error is zero. If no error is made, Bob prepares Alice's state with perfect fidelity. If Alice makes an error, there is still some overlap with the correct state, given by $\sin^2\theta$. The fidelity is therefore

$$F=(1-P_e)+P_e\sin^2\theta=1-\tfrac{1}{2}(1-\cos\theta)\cos^2\theta. \quad (3)$$

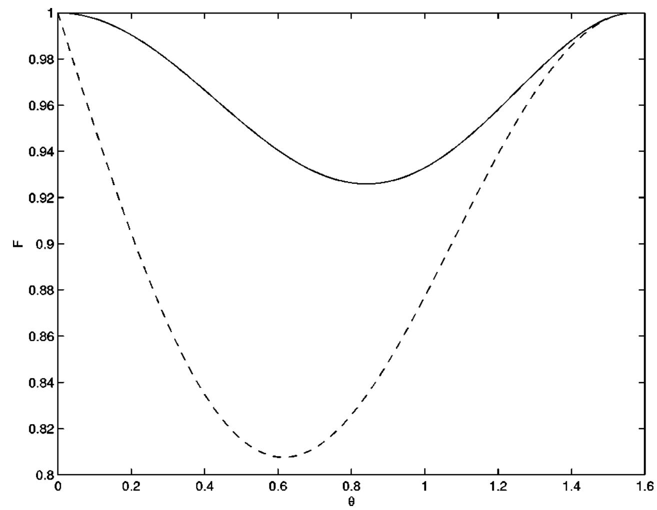

FIG. 1. Fidelity when Bob guesses one of the two input states. The solid line corresponds to the measurement that minimizes Alice's probability of error, Eq. (3), and the dashed line to a measurement giving unambiguous discrimination of the two states, Eq. (4).

For orthogonal states, $\theta=0$, $F_1=1$. For maximally nonorthogonal states with $\theta=\pi/4$, $F_1=0.927$.

An alternative strategy is to construct a POVM that distinguishes the two outcomes $|\psi_1\rangle$ and $|\psi_2\rangle$ with no probability of error, but that has a third outcome where the state is completely unknown. Then the maximum probability of a successful outcome is [7–9], which is $P_s=1-\sin\theta$ in our case. If the ''don't know'' outcome is obtained, Bob chooses at random which state to prepare. In half the cases, he succeeds. If he fails, there is still an overlap with the correct state. Therefore the fidelity is

$$F=1-\tfrac{1}{2}\sin\theta+\tfrac{1}{2}\sin^3\theta. \quad (4)$$

This fidelity is always lower than that achieved by minimizing the probability of error (see Fig. 1).

However, the strategy where Alice minimizes her probability of error and Bob prepares one of the input states is not optimal. It is possible to achieve a higher fidelity if Bob prepares a guess that has a slightly higher overlap with the other state to take into account the possibility that Alice makes an error. Alice still makes the measurement that minimizes her probability of error. For the two states $|\psi_1\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$ and $|\psi_2\rangle=\sin(\theta/2)|0\rangle+\cos(\theta/2)|1\rangle$ this is a projection onto $|0\rangle$ or $|1\rangle$. The positive operators of the POVM to be performed are $A_1=|0\rangle\langle 0|$ and $A_2=|1\rangle\langle 1|$, and the corresponding probabilities are

$$p(1|\psi_1)=|\langle 0|\psi_1\rangle|^2=\cos^2\left(\frac{\theta}{2}\right),$$

$$p(2|\psi_1)=|\langle 1|\psi_1\rangle|^2=\sin^2\left(\frac{\theta}{2}\right),$$

$$p(1|\psi_2)=|\langle 0|\psi_2\rangle|^2=\sin^2\left(\frac{\theta}{2}\right),$$

$$p(2|\psi_2)=|\langle 0|\psi_2\rangle|^2=\cos^2\left(\frac{\theta}{2}\right).$$

The fidelity is

$$F_{cl}=\tfrac{1}{2}[p(1|\psi_1)|\langle\alpha|\psi_1\rangle)|^2+p(2|\psi_1)|\langle\beta|\psi_1\rangle|^2 +p(1|\psi_2)|\langle\alpha|\psi_2\rangle|^2+p(2|\psi_2)|\langle\beta|\psi_2\rangle|^2],$$

where $|\alpha\rangle$ and $|\beta\rangle$ are Bob's guessed states. Assuming that the fidelity must be the same under an interchange of the two states, and that the guessed states share the same symmetry as the input states, so that $|\langle\alpha|\psi_1\rangle|^2=|\langle\beta|\psi_2\rangle|^2$, and $|\langle\beta|\psi_1\rangle|^2=|\langle\alpha|\psi_2\rangle|^2$, the fidelity becomes

$$F_{\rm cl}=p(1|\psi_1)|\langle\alpha|\psi_1\rangle|^2+p(2|\psi_1)|\langle\beta|\psi_1\rangle|^2, \tag{5}$$

$$=\cos^2\frac{\theta}{2}\cos^2\left(\frac{\theta-\alpha}{2}\right)+\sin^2\frac{\theta}{2}\sin^2\left(\frac{\theta+\alpha}{2}\right).$$

Differentiating with respect to the choice of guessed angle α gives

$$\frac{\partial F_{\rm cl}}{\partial\alpha}=p(1|\psi_1)\sin(\theta-\alpha)+p(2|\psi_1)\sin(\theta+\alpha).$$

We find the maximum value of $F_{\rm cl}$ by setting $(\partial F_{\rm cl}/\partial\alpha)=0$. The angle that gives a maximum is

$$\alpha=\tan^{-1}\left(\frac{\sin\theta}{\cos^2\theta}\right).$$

Substituting into Eq. (5) gives the fidelity plotted in Fig. 2. Notice that this fidelity, unlike the fidelity of the other strategies, is symmetrical about $\theta=\pi/4$. This result coincides with the following expression derived by Fuchs and Peres [10], in the context of eavesdropping:

$$F_{\rm cl}=\frac{1}{2}(1+\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2+|\langle\psi_1|\psi_2\rangle|^4}).$$

In this scenario, Alice tries to communicate to Bob one of a set of nonorthogonal states, which is intercepted by Eve. Eve wants to extract as much information as possible from a measurement on the state, and at the same time to prepare a new state with as high a fidelity as possible with Alice's original state so as to deceive Bob. Eve performs the dual function of Alice as measurer and Bob as preparer in our scheme, where Alice and Bob are connected only by a classical channel. It is plausible that for Bob to maximize the fidelity, he should have maximum information about the state and that Alice should also maximize her information by making the measurement that minimizes the probability of error. This suggests that the fidelity of Eq. (5) is optimal. The symmetry about $\theta=\pi/4$ may indicate optimality since all the less efficient strategies that we investigated do not possess this symmetry. Fuchs and Peres give further numerical and plausibility arguments in support of the optimality of this fidelity.

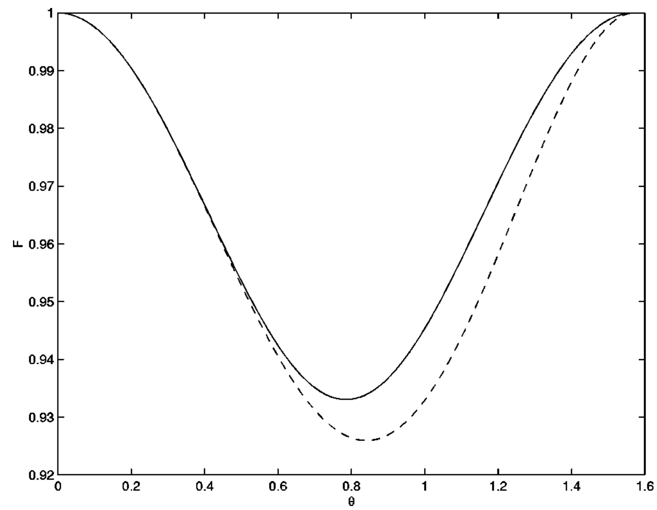

FIG. 2. Fidelity when Alice minimizes her probability of error. The dashed line shows the case where Bob prepares the state she specifies, Eq. (3), and the solid line the case where he optimizes his guess, Eq. (5).

Until now, the discussion has focused on the situation in which Alice and Bob communicate only by a classical channel. We now consider how shared entanglement can improve the fidelity of teleportation.

III. USE OF ENTANGLEMENT

If Alice and Bob share only one entangled pair, perfect two-state teleportation cannot be achieved without a full unit of entanglement. By contrast, in the asymptotic case with many copies of the state and many entangled pairs, perfect teleportation may be achieved with less than one full unit of entanglement for each qubit communicated.

A. Single-channel case

We prove that it is not possible to teleport perfectly with less than one full unit of entanglement, even if the state to be teleported comes from a known ensemble of only two nonorthogonal states. Let the state to be teleported be $|\phi\rangle_1$, and the entangled channel $|\phi\rangle_{23}$. Then the initial state of the three particles may be written as

$$|\phi_1|\psi\rangle_{23}=\sum_k c_k^{\phi}|k\rangle_{12}U_k^{-1}|\phi\rangle_3,$$

where the coefficient c_k^{ϕ} may depend on the initial state ϕ. The state has been expanded as a bipartite decomposition of the first two particles versus the third, where the orthonormal basis of the first two particles is given by $\{|k\rangle_{12}\}$, and the corresponding states of the third particle are $U_k^{-1}|\phi\rangle_3$, not necessarily orthogonal. Any general teleportation scheme must be of this form. The state can be transformed unitarily as

$$U(|\phi\rangle_1|\psi\rangle_{23})=\left(\sum_k c_k^{\phi}|k\rangle_{12}\right)|\phi\rangle_3 \tag{6}$$

by the controlled unitary operation U_k on the third particle. Let $|A(\phi)\rangle_{12}=(\Sigma_k c_k^{\phi}|k\rangle_{12})$ and consider two input states, $|\phi\rangle$ and $|\phi'\rangle$. By taking the overlap of Eq. (2) with a similar equation for $|\phi'\rangle$, we obtain

$${}_1\langle\phi'|\phi\rangle_1=[{}_{12}\langle A(\phi')|A(\phi)\rangle_{12}]({}_3\langle\phi'|\phi\rangle_3).$$

Since

$${}_1\langle\phi'|\phi\rangle_1={}_3\langle\phi'|\phi\rangle_3,$$

it follows that either $\langle\phi'|\phi\rangle=0$, or ${}_{12}\langle A(\phi')|A(\phi)\rangle_{12}=1$. If $\langle\phi'|\phi\rangle=0$, the two input states are orthogonal and perfect teleportation can be achieved without the use of any entanglement at all, since an exact measurement to distinguish the states can be performed. The vectors $|A(\phi)\rangle_{12}$ and $|A(\phi')\rangle_{12}$ are normalized. Hence if ${}_{12}\langle A(\phi')|A(\phi)\rangle_{12}=1$, then $|A(\phi)\rangle_{12}=|A(\phi')\rangle_{12}$ and consequently the coefficients c_k^{ϕ} must be independent of the input state ϕ, so that $c_k^{\phi}=c_k^{\phi'}$. Therefore the probability of obtaining the result k is independent of the input state.

Any state to be teleported can be written as a linear combination of the states $|\phi\rangle$ and $|\phi'\rangle$:

$$|\psi\rangle=a|\phi\rangle+b|\phi'\rangle.$$

If both $|\phi\rangle$ and $|\phi'\rangle$ can be teleported perfectly by the same operation, there exists a unitary transformation U such that

$$U(|\phi\rangle_1|\psi\rangle_{23})=\left(\sum_k c_k|k\rangle_{12}\right)|\phi\rangle_3$$

and

$$U(|\phi'\rangle_1|\psi\rangle_{23})=\left(\sum_k c_k|k\rangle_{12}\right)|\phi'\rangle_3,$$

where we have shown that the coefficients c_k do not depend on the input state. Therefore

$$U(a|\phi\rangle_1+b|\phi'\rangle_1)|\psi\rangle_{23}=\sum_k c_k|k\rangle_{12}(a|\phi\rangle_3+b|\phi'\rangle_3),$$

and so any state can be teleported perfectly. This would mean it is possible to perfectly teleport a maximally mixed state. By the arguments of the Introduction, this would require a full unit of entanglement.

B. Asymptotic case

Alice's qubit is an equally weighted mixture of the two possible input states and so can be described by the density matrix

$$\rho=\tfrac{1}{2}(|\psi_1\rangle\langle\psi_1|+|\psi_2\rangle\langle\psi_2|).$$

Now if Alice possesses a large number n of copies of the qubit, she may use Schumacher compression [11] to compress the same information into $nS(\rho)$ qubits, where $S(\rho)=-\mathrm{tr}(\rho\log_2\rho)$ is the Von Neumann entropy of the qubit ρ. If $\theta=0$ and the two states are orthogonal, $S(\rho)=\log_2 2=1$. This is the only case where no compression is possible. For two maximally nonorthogonal states, with $\theta=\pi/4$, $S(\rho)\approx 0.907$ and transmission requires 0.907 ebits per qubit of information.

If Alice and Bob share a large number m of nonmaximally entangled pairs in the state ρ_{AB}, with $\rho_A=\mathrm{Tr}_B(\rho_{AB})$, they may distill $mS(\rho_A)$ maximally entangled pairs using only local operations and classical communication [12,13]. The quantity $S(\rho_A)$ denotes the amount of entanglement in the shared pairs, and for a maximally entangled state, $S(\rho_A)=1$. The amount of entanglement $S(\rho_A)$ required per qubit of information sent by Alice is $S(\rho_A)=(n/m)S(\rho)$, which may be less than 1 in the limit of large m and n, when the input states are nonorthogonal. Clearly then, the asymptotic case is different from the situation where only single copies of the states are available.

IV. TELEPORTATION THROUGH A NONMAXIMALLY ENTANGLED CHANNEL

Given that when Alice and Bob share only one nonmaximally entangled channel it is not possible to perform two-state teleportation perfectly, we would like to know how high a fidelity can be achieved. Below, we compare several different strategies; however, it is still an open question what the most optimal scheme would be.

If we apply the standard teleportation procedure, sending the initial state

$$|\psi\rangle_1=\cos\frac{\theta}{2}|0\rangle+\sin\frac{\theta}{2}\exp(i\phi)|1\rangle$$

through the nonmaximally entangled channel

$$|\psi\rangle_{23}=\alpha|00\rangle+\beta|11\rangle,$$

then the initial state of the three particles may be written as

$$\begin{aligned}|\psi\rangle_{123}=\frac{1}{\sqrt{2}}\Bigg[&|\phi^+\rangle\left(\alpha\cos\frac{\theta}{2}|0\rangle+\beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle\right)\\&+|\phi^-\rangle\left(\alpha\cos\frac{\theta}{2}|0\rangle-\beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle\right)\\&+|\psi^+\rangle\left(\alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle+\beta\cos\frac{\theta}{2}|1\rangle\right)\\&+|\psi^-\rangle\left(-\alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle+\beta\cos\frac{\theta}{2}|1\rangle\right)\Bigg].\end{aligned} \tag{7}$$

Without loss of generality, we assume that α and β are real and that $\alpha\leqslant\beta$. The fidelity is given by

$$F(|\psi\rangle)=\sum_{i=1}^{4} p(i|\psi)|\langle\psi|\alpha_i\rangle|^2,$$

where i is the index of the projections $A_i=|\phi_i\rangle\langle\phi_i|$ onto the four Bell states

$$|\phi_1\rangle=|\phi^+\rangle,$$

$$|\phi_2\rangle=|\phi^-\rangle,$$

$$|\phi_3\rangle=|\psi^+\rangle,$$

$$|\phi_4\rangle=|\psi^-\rangle,$$

and $|\alpha_i\rangle$ is Bob's normalized and corrected outcome $|\alpha_i\rangle$ given the measurement result i. The probability of Alice measuring $|\phi^+\rangle$ or $|\phi^-\rangle$, given the input state $|\psi\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)\exp(i\phi)|1\rangle$, is

$$p(1|\psi)=p(2|\psi)=\frac{1}{2}\left(\alpha^2\cos^2\frac{\theta}{2}+\beta^2\sin^2\frac{\theta}{2}\right);$$

the probability of measuring $|\psi^+\rangle$ or $|\psi^-\rangle$ is

$$p(3|\psi)=p(4|\psi)=\frac{1}{2}\left(\alpha^2\sin^2\frac{\theta}{2}+\beta^2\cos^2\frac{\theta}{2}\right).$$

The fidelity is then

$$F(|\psi\rangle)=\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}+\alpha\beta\sin^2\theta.$$

Averaged over all initial states, this gives

$$F_{\rm av}=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}+\alpha\beta\sin^2\theta\right)\sin\theta\, d\theta\, d\phi$$
$$=\tfrac{2}{3}(1+\alpha\beta). \tag{8}$$

It can be shown, using a result of Horodecki *et al.* [14], that the average fidelity given in Eq. (8) is optimal for any teleportation scheme, whatever Alice's measurement or Bob's corrections. Horodecki *et al.* derive a general relation between the optimal fidelity of teleportation $F_{\rm tele}$ and the maximal singlet fraction f, defined below, of the state used for teleportation:

$$F_{\rm tele}=\frac{2f+1}{3}.$$

For the nonmaximally entangled state $\alpha|00\rangle+\beta|11\rangle$, the maximal singlet fraction is

$$f=\left|\frac{1}{\sqrt{2}}(\langle 00|+\langle 11|)(\alpha|00\rangle+\beta|11\rangle)\right|^2=\tfrac{1}{2}(1+2\alpha\beta),$$

and hence the optimal fidelity of teleportation is given by Eq. (8).

In the two-state case, where Alice has either $|\psi_1\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$ or $|\psi_2\rangle=\sin(\theta/2)|0\rangle+\cos(\theta/2)|1\rangle$ with equal probabilities, the fidelity is

$$F=\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}+\alpha\beta\sin^2\theta. \tag{9}$$

When $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, the fidelity can only be unity if the channel is maximally entangled, $\alpha=\beta=1/\sqrt{2}$.

Another strategy for teleportation is based on first purifying the channel. Purification has some probability of converting the state to a maximally entangled state, which can achieve perfect teleportation, and some probability of failing, so that no entanglement is shared and Alice and Bob must revert to the classical methods for sending the state with no shared entanglement. For a single copy, the best purification is the "procrustean" method [12], which has optimal efficiency $2\alpha^2$ [15]. When the purification fails, Alice and Bob are left with a product state. The input state is unaffected by purification, so Alice may employ the best strategy for transmitting the state without entanglement. For a completely unknown input state, the fidelity is $F_{cl}=\frac{2}{3}$; hence the fidelity is

$$F=\tfrac{2}{3}(1+\alpha^2). \tag{10}$$

Higher fidelities are achieved in the two-state case. Then the best fidelity that may be achieved is

$$F=2\alpha^2+(1-2\alpha^2)F_{cl}, \tag{11}$$

where

$$F_{cl}=\cos^2\frac{\theta}{2}\cos^2\left(\frac{\theta-\alpha}{2}\right)+\sin^2\frac{\theta}{2}\sin^2\left(\frac{\theta+\alpha}{2}\right)$$

is the best measurement strategy with no entanglement with

$$\alpha=\tan^{-1}\left(\frac{\sin\theta}{\cos^2\theta}\right).$$

For a completely unknown input state, teleporting directly through the nonmaximally entangled channel is always better than the strategy based on purification, Eq. (10), since $\alpha\leqslant\beta$; see Fig. 3. For two input states, on the other hand, the fidelities of the different methods are plotted in Fig. 4. The direct method is no longer always better than the purification method, though it is better when the entanglement in the channel is high, in which case it approaches the average

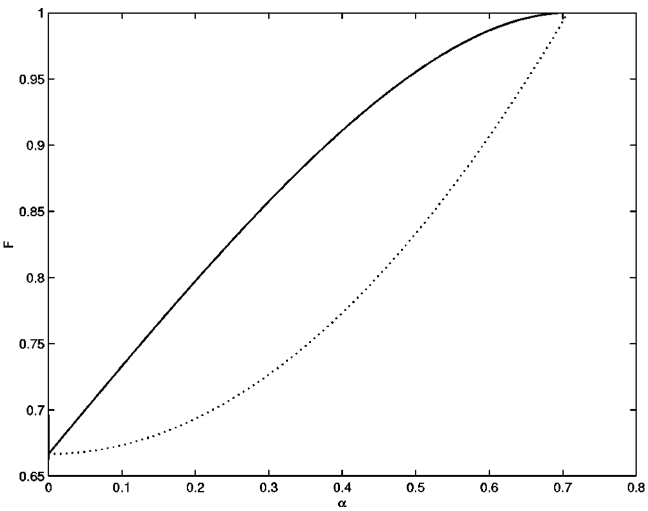

FIG. 3. Teleportation through a nonmaximally entangled channel for an unknown state. The dotted line shows the purification method, Eq. (10), the solid line the direct method, Eq. (8).

fidelity. For low entanglement, the efficiency of the direct method falls off steeply and becomes even worse than the classical strategy without entanglement.

For a completely unknown state, teleportation via either strategy is always better than the classical method of measuring and communicating the result. However, when there are only two possible input states, a large amount of information may be gained just by Alice measuring the state she has. It turns out that the fidelity that may be achieved by Alice measuring her state and telling Bob the result classically is higher than would result from a direct teleportation, if the channel has low entanglement. On the other hand, when the channel is first purified, it is possible to take advantage of the high classical fidelity by employing the classical strategy when the purification fails. It is possible to do this because it is known when the purification has failed. Hence in the two-state case, the purification method is better for low entanglement than the direct method.

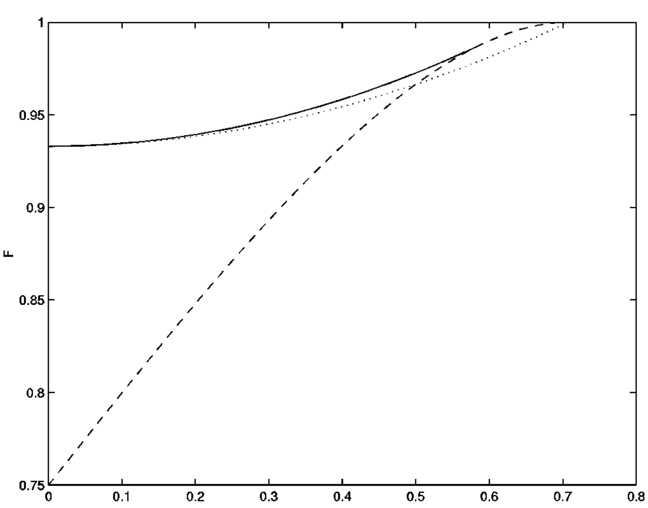

FIG. 4. Teleportation through a nonmaximally entangled channel for two states with $\theta=\pi/4$. The dotted line shows the purification method, Eq. (11), the dashed line the direct method using the standard corrections, Eq. (9), and the solid line the optimal combination of the two methods, Eq. (12).

In the two-state case, it is not known what the optimal teleportation scheme is. The best bound we have found is based on a combination of the direct and purification methods. This may be achieved by Alice partially purifying the entangled channel $\alpha|00\rangle+\beta|11\rangle$ to a more entangled channel, $\alpha'|00\rangle+\beta'|11\rangle$, where $\alpha'\geqslant\alpha$. The probability of succeeding in this purification is $P_s=(\alpha/\alpha')^2$. If the purification succeeds, the direct method may be employed on the more entangled state. If it fails, the best classical strategy must be employed. Hence the fidelity is given by

$$F=\left(\frac{\alpha}{\alpha'}\right)^2 F_{\mathrm{dir}}(\alpha')+\left[1-\left(\frac{\alpha}{\alpha'}\right)^2\right]F_{\mathrm{class}}. \tag{12}$$

For a particular nonmaximally entangled channel α, this fidelity is maximized by purifying to a particular channel characterized by α'.

V. RELATION TO TELECLONING

Limitations on the fidelity of teleportation can be related to limitations on the fidelity of cloning nonorthogonal quantum states. When a perfect teleportation is achieved, there should be no information about the state left on Alice's side that would enable her to construct any approximate copy of the state in addition to the perfectly teleported state. Teleportation using a maximally entangled pair achieves perfect fidelity, and the measurement on Alice's side provides no information since the probability of obtaining the different measurement outcomes is independent of the input state. This was also indicated by Nielsen and Caves [16], who showed that teleportation is a special case of reversing a quantum measurement, and that a necessary condition for reversibility of a general quantum operation is that no information about the prior state be obtainable from the measurement. On the other hand, if the channel is not maximally entangled, perfect teleportation cannot be achieved and Alice's measurement may provide some information about the input state. We have seen that when there is no entanglement in the channel at all, the optimal strategy is for Alice to extract as much information as possible from her measurement. The measurement result may then be used to prepare an arbitrary number M of identical imperfect copies of the original state with fidelity given by Eq. (1). This type of cloning has been called ''classical cloning'' [17] to distinguish it from the more general operation of quantum cloning that is based on unitary evolution of the input with an ancilla. Quantum cloning can achieve higher fidelities than classical cloning for a finite number of copies M. The process of quantum cloning allows the use of more entanglement than classical cloning since the ancilla may remain entangled to the input and the copies, which may also be entangled to one another. For two-state teleportation through a nonmaximally entangled channel, there is a trade off between the classical cloning based on directly measuring the input state and the fidelity that can be achieved by teleportation based on the entanglement. The exact relation between the constraints on sharing information among copies in cloning and in teleportation is a topic for further research. However, one way in which the relationship between cloning and teleportation may be pursued is through a combination of the two procedures in ''telecloning.'' We now investigate the effect of *a priori* information on this protocol.

A. State-dependent telecloning

Teleportation has been combined with optimal universal cloning from 1 to M copies [4]. This is achieved by performing the usual teleportation protocol but with the entangled channel being a multiparticle entangled state, called the ''telecloning'' state. For $M=2$, the telecloning state is a four-qubit state

$$|\psi_{TC}\rangle=\frac{1}{\sqrt{2}}(|0\rangle|\phi_0\rangle+|1\rangle|\phi_1\rangle), \tag{13}$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the optimal cloning states produced by acting with the optimal cloning transformation U_{12} on $|0\rangle$ and $|1\rangle$, respectively,

$$|\phi_0\rangle=U_{12}(|0\rangle_A|00\rangle)=\sqrt{\tfrac{2}{3}}|0\rangle_A|00\rangle+\sqrt{\tfrac{1}{6}}|1\rangle_A(|01\rangle+|10\rangle),$$

$$|\phi_1\rangle=U_{12}(|0\rangle_A|10\rangle)=\sqrt{\tfrac{2}{3}}|1\rangle_A|11\rangle+\sqrt{\tfrac{1}{6}}|0\rangle_A(|01\rangle+|10\rangle),$$

where subscript A denotes the ancilla. In the telecloning state, the first two qubits and the ''port'' are held by Alice and the last two qubits belong to two distant users, Bob and Claire. When the other qubits are traced over after telecloning, these yield the optimal clones of Alice's input state. The total amount of entanglement between Alice and the other users, given by the Von Neumann entropy of the reduced density matrix after tracing over one side, was found to be $\log_2(3)$, clearly less than the two units of entanglement required if cloning is performed first and followed by the standard teleportation.

Adapting the telecloning scheme to the communication of two states produces a surprising result in terms of the amount of entanglement required. Bruss *et al.* [18] have found the optimal cloning transformation U with respect to the global fidelity for two-state cloning from one copy to two. An ancilla is not necessary. Following the same procedure as in the universal case for constructing the telecloning state, we may add an ancilla, giving the cloned states to be

$$|\phi_0\rangle=U_{12}(|0\rangle_A|00\rangle)=a|0\rangle_A|00\rangle+b|1\rangle_A(|01\rangle+|10\rangle)+c|0\rangle_A|11\rangle,$$

$$|\phi_1\rangle=U_{12}(|0\rangle_A|10\rangle)=c|1\rangle_A|00\rangle+b|0\rangle_A(|01\rangle+|10\rangle)+a|1\rangle_A|11\rangle,$$

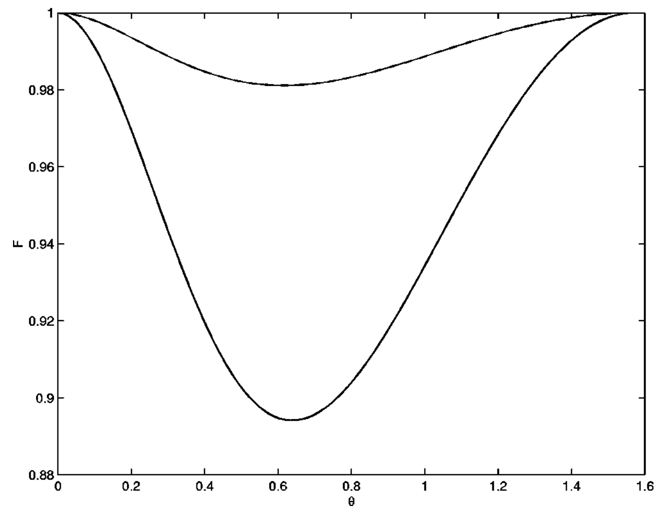

FIG. 5. The global fidelity of the clones produced in the telecloning scheme (solid line) as compared to the optimal global fidelity for two-state cloning (dotted line).

where a, b, and c depend on the overlap of the two states, as given in the paper [18]. The telecloning state is constructed just as before, Eq. (13). The ancilla is required in order that the recipients may use the standard Pauli rotations to correct their state after they receive the classical message from Alice. Notice, however, that $|\phi_0\rangle$ and $|\phi_1\rangle$ are no longer the optimal clones. The fidelity of cloning is shown in Fig. 5.

The reduced density matrix found by tracing the density matrix for the telecloning state over Alice's two qubits is

$$\rho_{34}=\frac{1}{2}\begin{pmatrix} a^2+b^2+c^2 & 0 & 0 & 2a(b+c) \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 2a(b+c) & 0 & 0 & a^2+b^2+c^2 \end{pmatrix}.$$

The entanglement between the two sides now increases with the overlap of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$, but is always

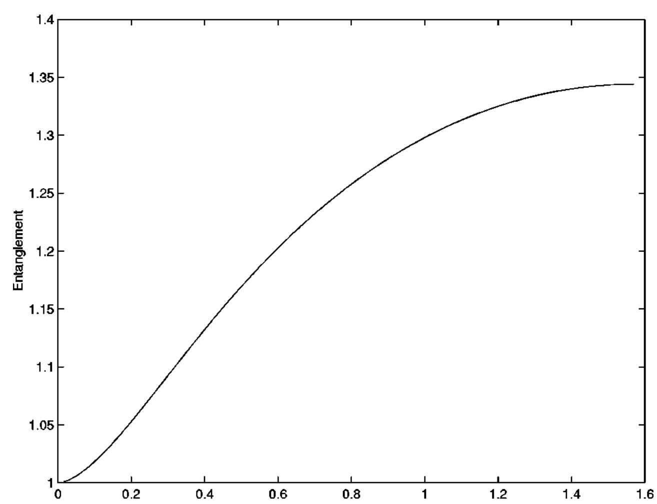

FIG. 6. Entanglement between Alice and receivers in telecloning.

less than $\log_2(3)\approx 1.585$; see Fig. 6. However, each qubit is maximally mixed so the entanglement between any one qubit and the other three is 1. This means that Alice's port qubit does share a unit of entanglement with the other three qubits. This is consistent with the requirement that perfect teleportation of two states employ a full unit of entanglement. In this telecloning scheme, the amount of overall entanglement is lower than in the universal case. It is an interesting question whether a telecloning scheme giving the optimal two-state cloning fidelity would also require less entanglement.

VI. CONCLUSION

In this paper, we have shown the surprising result that *a priori* knowledge makes no difference to the amount of entanglement required for perfect teleportation. We have computed lower bounds for two-state teleportation fidelity using a nonmaximally entangled pure state as a channel, and the exact result for the two-state fidelity with no entanglement.

This work opens a number of possible directions for future research. In this paper, only pure entangled states were considered as channels for teleportation. The investigation could be extended to include mixed entangled states. The relationship between cloning and teleportation with *a priori* knowledge could be investigated further by finding the amount of entanglement required by a state-dependent telecloning scheme that preserves the optimality of the clones produced. Asymmetric telecloning or general N to M state-dependent telecloning could also be considered. It may be possible to exactly quantify the relationship between the amount of information Alice gains from her measurement, the amount of entanglement in the channel, and Bob's information. Our work provides a different way of understanding the respective roles of classical information and quantum entanglement in the new field of quantum information processing.

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